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## I. The Problem

The construction of single sex population models on the assumption that the distribution of the other sex had to adjust its numbers accordingly (Kendall, 1949), has a long history despite time old objections (Karmel, 1947, 1948) that results based on one sex model are not consistent with the other. Recently, simplicity and operational convenience of single sex models are not regarded as sufficient justifications for their use, so that the population analysts in general, and the model builders in particular, are putting greater and greater emphasis on the effects of the differences in the age structure of the two sexes on such models. For some time in the past, these researchers are experimenting with different functions of age and sex composition of population, as well as a set of consistency conditions which these functions must meet, in order to arrive at one or more functions that can be regarded as appropriate for the measurement of fertility. Recent attempts (Feeney, 1972; Keyfitz, 1968, 1971; McFarland, 1970; Partlett, 1972 and Pollard, 1971) to develop what is now known as a marriage function are frustrated by the inability of the proposed functions to meet the necessary consistency conditions and the search is still going on. Das Gupta (1972) used the name 'effective population' to propose another such function which he defined without the explicit acknowledgement of marriage as a social phenomenon. This function does not, and is not required to meet the consistency conditions of the marriage function outlined by other researchers (McFarland, 1972; Partlett, 1972), and has its own set of constraints. The functional form advanced by him (1973) is

$$P(a,a',t) = P(t) - \frac{M(a,t)F(a',t)}{M(t)F(t)}$$
(1)

where

- P(a,a',t) is the effective population corresponding to males aged a and females aged a' at time t,
- M(a,t) and F(a,t) are the numbers of males aged a and females aged a' respectively at time t,
- M(t) and F(t) are the total male and female populations at reproductive ages at time t, and

$$P(+) = M(+) + F(+).$$

Das Gupta then defined B(a,a',t) as the births corresponding to men aged a and women aged a' at time t and the ratio of B(a,a',t)to P(a,a',t) as the two-sex-age-specific birth rate m(a,a',t). He was able to show that when m(a,a',t) is independent of t, there will be a single intrinsic rate of growth for the two sexes. In that case, the age old problems created by the different intrinsic rates of the one-sex models and others caused by the unrealistic restrictions that the parameters of one sex must meet in order to be consistent with those of the other, ought to disappear.

In his illustrations with US data, Das Gupta observed that his intrinsic rate of growth does not always lie in the interval generated by the two rates of one-sex models, a condition that was suggested (Coale, 1972) as desirable. Although, such a requirement cannot be mathematically established, he seems to admit the importance of this condition. He has also acknowledged the desirability of making the effective population dependent on the age difference of the two sexes, as is customary for a marriage function and which is not met by P(a,a',t).

Be that as it may, it is easy to see that his definition of an effective population is an abstraction in the sense that the population comprising P(a,a',t) is firstly, not identifiable and secondly, another form of the function, if it exists, will in all likelihood, result in a different intrinsic rate of growth.

## 2. Birth Function

At this point, it seems important to point out that the possibility of defining a two-sex model as an explicit function of age of any one of the two sexes need not be overlooked. For example, B(a,t) the number of male births to male population aged a at time t, namely M(a,t), can be regarded as proportional to M(a,t), to the proportion of female to the total population in the reproductive ages, and finally to another function of age and time. Specifically, the form proposed is

$$B_{M}(a,t) = K_{M}(a,t) - \frac{M(a,t)F(t)}{P(t)}$$
(2)

which, to some extent, resembles Pollard's (ibid) definition of a marriage function, ignoring the age of the females. The traditional male age-specific birth rate can then be expressed as

$$m_{M}(a,t) = \frac{B_{M}(a,t)}{M(a,t)} = K_{M}(a,t) \frac{F(t)}{P(t)} (3)$$

Similarly, age-specific rates for female births to females aged a' can be written as

$$m_{F}(a',t) = K_{F}(a',t) \frac{M(t)}{P(t)}$$
 (4)

The proportions of male and female to total reproductive population appearing as factors in (2) and (4) respectively take care, in a simple way, the contribution of the other sex towards the reproductive activity of the population. Inconsistencies in age-sex compositions, like the absence of one sex corresponding to some age or ages in the reproductive intervals are not relevant from the point of view of stability, and therefore, can be ignored. For an approach towards stability, it is somewhat imperative that all ages within the reproductive interval for both sexes be adequately represented at the initial time period and thereafter, so that the fertility rates can continue to maintain non zero values for any length of time. As such,  $K_M(a,t)$  or  $K_F(a',t)$  can account for the

variations in  $m_M(a,t)$  or  $m_F(a',t)$  either due

to changes in reproductive behavior or due to some abnormal conditions or due to both. The implication of the assumption of  $K_M(a, +)$  =

assumption of constancy in the patterns of age-sex-specific mortality rates are examined next. It may be noted that given the constancy of the K values, the fertility rates of one sex become directly proportional to the ratio that the size of the reproductive population of the other sex bears to that of the two sexes.

As yet, the variables F(t)/P(t) and M(t)/P(t) have not been made explicit. It may be pointed out that these variables should only meet the logical requirement P(t) = M(t) + F(t), so that the two fractions can add up to one. However, M(t) and F(t) need not be equal to the sum of all the males and females in their respective reproductive intervals, although, as will be shown later (Section 5), those values have been used to derive their initial and intrinsic estimates for operational simplicity. In fact, from a mathematical point of view, the multipliers of  $K_M(a,t)$  and

 $K_{c}(a',t)$  in (3) and (4) should be two fractions,

say, u(t) and v(t) so that u(t) + v(t) = 1 for

all t and further that  $\Sigma M(a,t)m_M(a,t)$  be

proportional to  $\Sigma F(a',t)m_F(a',t)$  where the

constant of proportionality is the sex ratio at birth. In practice, however, one may assume that the variables  ${\rm K}_{\rm M}({\rm a})$  and  ${\rm K}_{\rm F}({\rm a}^{\,\prime})$  will be

uncorrelated with M(a,t) and F(a',t) respectively. In that case, the use of  $\Sigma$ M(a,t) and  $\Sigma$ F(a',t) for M(t) and F(t) will provide close approximations of u(t) and v(t). It may be mentioned that the actual values of u(t) and v(t), rather than their estimates, may be used if birth estimates are necessary for t>0. However, for considering the implications of long term effect of the constancy of the K values, u(t) and v(t) need not be explicitly obtained beyond the initial time periods as has been shown next.

Let t be the number of years past the initial year such that t is greater than the last age of the reproductive interval of either sex. In that case, the number of male births in year t, namely,

$$B_{M}(+) = \int_{0}^{\infty} B_{M}(+-a) p_{M}(a) m_{M}(a,+) da$$
 (5)

where  $p_{M}(a)$  is the probability of a male surviving from birth to age a. Because of (3),

equation (5) can be written as

$$\frac{B_{M}^{(+)}}{\int_{0}^{\infty} B_{M}^{(+-a)} g_{M}^{(a)} da} = \frac{F(+)}{P(+)}$$
(6)

where

$$g_{M}(a) = p_{M}(a)K_{M}(a)$$
 (7)

Combining similar expressions for the females with (6) one can write

$$\frac{B_{M}^{(+)}}{\int_{0}^{\infty}B_{M}^{(+-a)}g_{M}^{(a)}da} + \frac{B_{F}^{(+)}}{\int_{0}^{\infty}B_{F}^{(+-a')}g_{F}^{(a')}da'} = 1$$
(8)

Regarding the sex ratio at birth S as independent of time, so that  $B_M(t) = SB_F(t)$ , (8) can be simplified as

$$B_{F}^{(+)} \begin{bmatrix} 1 \\ \int_{0}^{\infty} B_{F}^{(+-a)} g_{M}^{(a)} da \\ \vdots \\ \int_{0}^{\infty} B_{F}^{(+-a')} g_{F}^{(a')} da' \end{bmatrix} = 1$$
(9)

Following the customary procedure of using a trial solution like

$$B_{F}(t-a) = B_{F}(t)e^{-ra}$$
 (10)

(9) is further reduced to

$$\int_{0}^{\infty} e^{-ra} g_{M}(a) da = \int_{0}^{\infty} e^{-ra'} g_{F}(a') da' = I \quad (II)$$

### 3. Examination of the Integral Equation

It is easy to see that there is only one real value of r that will satisfy (II). It can also be shown that this r will lie in the interval  $(r_M, r_F)$  where  $r_M$  and  $r_F$  are intrinsic rates of growth for the males and females respectively, obtained when  $m_M(a,t)$  and  $m_F(a',t)$  are assumed as independent of time together with the corresponding survivorship functions. That is to say,

$$m_{M}(a,+) = m_{M}(a,0) = K_{M}(a,0) - \frac{F(0)}{P(0)}$$
 (12)

and

$$m_F(a',+) = m_F(a',0) = K_F(a',0) - \frac{M(0)}{P(0)}$$
 (13)

for say, t=0.

The integral equation for  $r_M$ , namely,

can then be written as

$$\frac{1}{\int_{0}^{\infty} -r_{M}^{a} P_{M}(a) K_{M}(a) da} = \frac{F(0)}{P(0)}$$
(15)

since  $K_{M}(a)$  defined earlier is the same as  $K_{M}(a,0)$ . Adding similar expressions for the females to (15), one gets because of (7),

$$\frac{1}{\int_{0}^{\infty} e^{-r_{M}a} g_{M}(a)da} + \frac{1}{\int_{0}^{\infty} e^{-r_{F}a'} g_{F}(a')da'} = 1 (16)$$

an expression comparable to (II). Subtraction of (I6) from (II) produces

$$\frac{1}{\int_{0}^{\infty} e^{-ra}g_{M}(a)da} - \frac{1}{\int_{0}^{\infty} e^{-r}M^{a}g_{M}(a)da} = \frac{1}{\int_{0}^{\infty} e^{-r}F^{a'}g_{F}(a')da'} - \frac{1}{\int_{0}^{\infty} e^{-ra'}g_{F}(a')da'}$$
(17)

Because  $e^{-ra}$  decreases with increase in r for all values of a, it is clear that when l.h.s and therefore r.h.s of (17) are both positive  $r_M < r < r_F$  and similarly  $r_F < r < r_M$  when both sides of (17) are negative. In other words, the interval  $(r_F, r_M)$  includes r.

For the solution of (11), the following transformation

$$\int_{0}^{\infty} -ra -r\mu_{M} \int_{0}^{\infty} -r(a-\mu_{M}) g_{M}(a) da = e^{-r\mu_{M}} \int_{0}^{\infty} g_{M}(a) da \quad (18)$$

seems to be helpful where

$$\mu_{M} = \int_{0}^{\infty} ag_{M}(a) da / \int_{0}^{\infty} g_{M}(a) da$$
 (19)

is the average age of the curve of the net paternity function. Note from (7) and (3) that  $$r^{\infty}$$ 

$$g_{M}(a)da = R_{OM} \frac{P(0)}{F(0)}$$
 (20)

its power series and integrating, as

$$\int_{0}^{\infty} -ra = e^{-r\mu} R_{OM} + \frac{P(0)}{F(0)} (1+0(r^{2})) (21)$$

because of (19) and (20). Neglecting terms of  $O(r^2)$ , writing a similar expression for the females, taking reciprocals and substituting in (11), the equation

$$\frac{e^{\Gamma \mu}M}{R_{OM}P/F} + \frac{e^{\Gamma \mu}F}{R_{OF}P/M} = 1$$
 (22)

is obtained by dropping the time factor 0, which can be further approximated as

$$\frac{I+r\mu_{M}}{R_{OM}P/F} + \frac{I+r\mu_{F}}{R_{OF}P/M} = I$$
 (23)

The intrinsic rate for the two sexes can then be solved as a first approximation from

$$r = \frac{P(R_{OM}/F)(R_{OF}/M) - R_{OM}/F - R_{OF}/M}{\mu_{M}R_{OF}/M + \mu_{F}R_{OM}/F}$$
(24)

This solution of r can now be used to generate other intrinsic measures. Thus, the birth rates  $b_{M}$  and  $b_{F}$  for the males and the females are given by

$$b_{M} = -\frac{1}{\int_{0}^{\infty} e^{-ra} p_{M}(a) da},$$

$$b_{F} = \frac{1}{\int_{0}^{\infty} e^{-ra'} p_{F}(a') da'}$$
(25)

and

$$\frac{\overline{M}(+)b_{M}}{\overline{F}(+)b_{F}} = S$$
(26)

being the intrinsic ratio of male to female births where  $\overline{M}$  and  $\overline{F}$  stand for the total male and female populations respectively of all ages, in contrast to M and F that represent the same in reproductive age groups only. Therefore,

$$\frac{\overline{M}(+)}{\overline{F}(+)} = \frac{Sb_{\overline{F}}}{b_{\overline{M}}}$$
(27)

and further

$$\frac{\overline{M}(+)}{\overline{P}(+)} = \frac{Sb_F}{Sb_F+b_M}$$
(28)

$$\frac{\overline{F}(+)}{\overline{P}(+)} = \frac{b_{M}}{Sb_{F}+b_{M}}$$
(29)

The age-specific rates approach stability because of stability in the two age-structures and the identical rate of growth. Thus,

$$m_{M}(a,t) = m_{M}(a) = K_{M}(a) - \frac{F(t)}{P(t)}$$
 (30)

and

$$m_{F}(a') = K_{F}(a') \frac{M(+)}{P(+)}$$
 (31)

# 5. Application of Results

The table that follows is obtained for the year 1966 for US population and is based primarily on figures furnished by Das Gupta (<u>ibid</u>). For the computations of the various measures, the reproductive age intervals for the males and females are taken as 15-54 and 15-49 respectively since reproduction rates outside these intervals are extremely small.

TABLE I.	Intrinsic Rate of Growth and	Birth
Rate	of US Derived from Different	
	Models for the Year 1966	

Model	1000r		1000ь		
	Male	Female	Male	Female	
One Sex Two Sex (Das	12.7 15.1	9.5 15.1	22.9 24.5	19.2 23.7	
Gupta) Two Sex (Mitra)	12.5	12.5	22.8	20.8	

The year 1966 is chosen because this is one of the recent years in the time series generated by Das Gupta in which his intrinsic rate of growth for the two sex model falls outside the interval  $(r_{\rm M},r_{\rm F})$  and in fact, is

greater than the larger of the two rates. Differences in the intrinsic birth rates are also worth noting and all of these are indications of the extent to which such results can differ dependent as they are on the different assumptions underlying each one of them.

The intrinsic birth rates are next obtained from (30) and (31) in which the  $K_{\rm M}$  and  $K_{\rm F}$  values are derived from 1966 age-sex specific rates for five year age groups and the male and female populations in their respective reproductive age intervals. The ratio of their stable estimates is given by

$$\frac{M(t)}{F(t)} = S \frac{\int_{15}^{55} e^{-ra} p_{M}(a) da}{\int_{15}^{50} e^{-ra'} p_{F}(a') da'}$$
(32)

which is approximately equal to

$$S = \frac{L_{15}^{M} (1 - r \overline{x}_{M})}{\frac{L_{15}^{F}}{35} (1 - r \overline{x}_{F})}$$
(33)

neglecting terms of O(r<sup>2</sup>) and above. In (33),  $\overline{x}_{M}$  and  $\overline{x}_{F}$  are the average ages of males and females in their respective truncated stationary populations in which the sizes are

$$L^{M}$$
 and  $L^{F}$ . The results, shown in 40 15 30 15

Appendix Table I, are based on a sex ratio of births as 104.8 for the year 1966 corresponding to Das Gupta's proportion of male to female births  $\theta_{M}$ =.5118 and 1966 life tables (Keyfitz & Flieger 1971) which produced  $\overline{x}_{M}$ =34.5 and

The figures demonstrate the sensitive nature of the parametric relationships in a two-sex stable population model. Discrepancies caused by relatively large divergence in the rates of growth obtained separately for the two sexes (Table 1), can be completely removed through relatively small adjustments in the age-specific birth rates. This goes on to suggest that the initial age-specific rates for the two sexes are not always that inconsistent with one another, as they are often taken to be and as are also demonstrated by Das Gupta's model. It may also be pointed out that the ratio of the male to female population in their respective reproductive intervals observed at 1.05 in 1966 corresponds to an intrinsic value of 1.08 according to Das Gupta and 1.10 according to the present model. This is another indication of the

sensitivity of stable models mentioned earlier showing how a large imbalance in sex composition can be due to relatively smaller difference between the intrinsic and the observed age specific birth rates.

# 6. Summary and Concluding Observations

The idea of measuring reproduction rates specific for both sexes is gaining momentum over the past few years, and quite justifiably, attentions so far were focused on constructing appropriate population base specific for age combinations of both sexes. Researchers have come across insurmountable difficulties in their attempts to define such a function that would also satisfy certain logical consistency conditions appropriate for such measures. In this paper, an attempt has been made to define birth functions specific for one sex and age as dependent upon the composition of the reproductive population of both sexes. The specific rates so defined for the two sexes have next been incorporated in what may be regarded as a two-sex model. It has been shown that when the birth functions are proportional to the size of the population of a given sex and age and also vary directly with the proportion that the reproductive population of the other sex bears to that of the two sexes, the population can approach stability when both sexes also experience their specific mortality schedules. The intrinsic rates of growth for the two sexes become equal, and interestingly enough, this rate is found to lie in an interval bounded by the two similar intrinsic rates obtained from the two singlesex models.

# 7. References

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age group							Intrinsic Rates			
	m <sub>M</sub> (a,	m <sub>F</sub> (a', K <sub>M</sub> (	K <sub>M</sub> (a)	K <sub>M</sub> (a) K <sub>F</sub> (a')	Mi m <sub>M</sub> (a)	tra m <sub>F</sub> (a')	Das ( m*(a) M	Gupta . m*(a') F		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)		
15-19	11.2	34.4	.0230	.0671	11.0	35.1	10.5	38.1		
20-24	82.2	90.4	.1686	.1765	80.4	92.4	83.7	108.0		
25 <b>-</b> 29	93.4	72.5	.1915	.1415	91.3	74.1	103.2	87.4		
30-34	61.2	41.5	.1255	.0810	59.8	42.4	69,5	46.2		
35-39	34.9	20.5	.0716	.0400	34.1	20.9	37.9	20.4		
40-44	15.6	5.7	.0320	.0111	15.3	5.8	15.9	5.2		
45-49	6.1	0.4	.0125	.0008	6.0	0.4	6.0	0.3		
50-54	3.3		.0068		3.2		3.3			

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